PENTAGON-BASED RADIAL TILING WITH TRIANGLES AND RECTANGLES AND ITS SPATIAL INTERPRETATION

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Abstract: The paper considers a type of radial pentagon-based tiling consisting of two shapes: triangle and rectangle. The obtained solution has a spatial interpretation in a 3D arrangement of equilateral triangles and squares dictated by the particular array of concave cupolae of the second sort, minor type (*CC-II-5.m*). These cupolae are arranged so that their decagonal bases partly overlap, making a pentagonal pattern (similar to the one of the Penrose tiling). Covering the folds between the faces of such a polyhedral structure with polygons, we use exactly equilateral triangles and squares, thanks to the trigonometric properties of *CC-II-5.m*. Observed in the orthogonal projection onto the plane of the polygonal bases, this 3D "covering" is viewed as a pentagonal-based radial tiling in the Euclidean plane. Equilateral triangles will be projected into congruent isosceles triangles corresponding to those obtained by the radial section of a regular pentagon in 5 parts. The squares are projected into rectangles whose ratio is: $a:b = 1:\varphi/\sqrt{(1+\varphi^2)}$, where φ is the golden ratio. These triangles and rectangles form a radial tiling consisting of 5 sectors of the plane, where the patterns of the established tiles are repeated locally periodically. However, with 5-fold rotation of the pattern, the tiling itself is non-periodic. The various tiling solutions that can be obtained in this way may serve as inspiration for the geometric design, especially interesting in architecture and applied arts, e.g. for rosettes, brise soleils, mosaics, stained glass, fences, partition screens and the like.

Keywords: tiling, tile, pentagon, 5-fold symmetry, triangle, rectangle.

INTRODUCTION

The term "tiling" refers to the tessellation of the Euclidean plane with polygons. Polygons must be closed, convergent sections of the plane and connected to each other by their sides, without overlaps or gaps. These polygons are otherwise called "tiles" and can be regular or irregular. Also, the number of polygonal shapes participating in the tiling can be different, from the ones where they are all congruent, i.e. where the plane is tiled with one type of tile alone (monohedral), to those where more different polygons participate in the tiling (polyhedral).

The very word "tile" for a polygon and "tiling" for the process of covering a plane with such polygons, originates from practical and utility objects, such as ceramic tiles. As one of the oldest inventions of mankind, ceramics dates back to pre-Paleolithic times.[1] Tiling of surfaces, as part of the finishing works, more precisely interior and exterior decorations, according to some sources,[2] appears as early as the time of the Sumerian civilization, 4000 BC. We meet them in Mesopotamia, Ancient Egypt, Ancient Greece and Rome, Byzantium, Japan, China, among the indigenous Mesoamericans, and in the Arab world. Some of the iconic legacies of the Antiquity, which can be found in any relevant publication concerning the history of architecture and art, are adorned with ceramic tiles: Gate of the goddess Ishtar (ca. 575 BC, Babylon), geometric patterns in mosaics of Pompeii (79 AD), or arabesques that we encounter in mosques throughout the Islamic world, to mention only some.

It is understandable that (ceramic) tiles have been so widely used around the globe, and in almost all civilizations, due to the diverse possibilities of combining shapes and colors and the resulting most diverse decorative patterns, as well as due to evident durability and resistance to temperature and physical fractures.[3] Therefore, the practice of tiling surfaces in architecture and applied arts, especially in the interior is so common. Let us mention, apart from the most obvious tiling of floors and walls, mosaics, inlays and marquetry, also covering of roofs and even facades in certain parts of the world (e.g. Portugal).

Why is the geometric fitting of these tiles important? Not only because of the simpler continuation of the tiles to each other, since the geometry of the *prototiles* (shapes of the participating tiles) dictates the way of arranging them, but also because of the material savings by reducing cuts when the tiles overlap or by filling in gaps. Thus, the consumption of tiles is rational. Finally, there is a visual impression of rule and order, which one perceives as aesthetic, as opposed to chaotic and disorganized. The artistic approach allows more freedom in that, yet it adheres to the framework of the general rule.

The problem of geometric tiling is more demanding in terms of respecting the strict rules and methods of tilings origination, including also angular alignment, vertex figures, combinations of the prototiles, transformations and symmetries.

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1 The earliest pottery artifacts date from the period 29.000–25.000 BC, found in the Czech Republic, v. P. B. Vandiver et al., "The Origins of Ceramic Technology at DolniVĕstonice, Czechoslovakia", *Science*, 246 (4933), 1989, 1002–1008.

2 C. A. Pickover, *The Math Book: From Pythagoras to the 57th Dimension, 250 Milestones in the History of Mathematics*, Sterling, 2009, 372.

3 Thanks to the greater force required for greater deflection that will cause breaking stress.

TILING AS A GEOMETRIC PROBLEM

Tiling of the Euclidean plane, as a geometric problem, is considered to have been set by Kepler in his work *Harmonices Mundi*[4] in the early 16th century. Some sources[5] state that geometric dealing with the problem of tiling can be found in the works of Papus of Alexandria, 4th century A.D. while Dürer's works on this subject,[6] a century earlier, are also well known.

Kepler gave the basis for the theoretical study of tiling and gave a systematic approach to the problem, which is still being used as an indispensable source in this field. Centuries later, in the 20th century, scientists emerged who continued to investigate the geometric problem of tiling in a way that further pushed boundaries. Let us mention a few of the most important: M. Goodman-Strauss, M. Ghyka, K, Critchlow, B. Grünbaum and G. C. Shepard, D. Chavey, H.S.C. Coxeter, J. H. Conway, and R. Penrose. We lack the space to pay tribute to a multitude of others who have contributed to the topic, each from their own domain. Actually, the topic can be treated in a classical, constructive-geometric way, but in modern science, mathematical approaches are more common, which use discrete geometry, theory of finite groups or combinatorics as tools.

When arranging tiles in a compact tiling without gaps, the key is the shape of the selected tiles themselves – prototiles. Angular dimensions of the tiles dictate their layout. Yet, we can often form several different tilings with the same tiles, so the symmetrical relations between the tiles set the conditions for their arrangement. If we adopt the rule that tiles are also regular polygons, there are only 3 such tilings and they are regular (Platonic). If we allow the use of two different polygonal shapes in tiling so that all vertices, i.e. their vertex figures are identical, as is the case with the previous ones, we get 8 semi-regular (Archimedean) tilings. According to the systematization of Grünbaum and Shepard,[7] these are 1-uniform tilings, and consequently there are 2-uniform, or demi-regular, 3-uniform, 4-uniform and *k*-uniform tilings, where the number *k* denotes the number of different vertex figures. The number *k* can be infinite.[8]

All of the *k*-uniform tilings are periodic, which means: formed by translation of the *fundamental regions* into which the initial prototiles are grouped. Unlike them, there are also non-periodic tilings, where the translational repetition of the fundamental region is disturbed. The tiles and/or fundamental regions are arranged so that the whole plane cannot be tiled merely by translating them, although they can locally be arranged periodically. In this case, other plane transformations are needed for the formation of tiling, so rotational, reflective, radial, displaced radial[9] or spiral tilings

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4 J. Kepler, 1619. *Harmonices mundi. Libri V,* 1619, Reprint Culture et Civilisation, Bruxelles 1968.

5 D. P. Chavey, "Tilings by Regular Polygons II: A Catalog of Tilings", Computers & Mathematics with Applications, 17 (1–2), 1989, 147–165.

6 A. Dürer, *Unterweysung der messungmitdem Zirckel und Richtscheyt*, 1st Edition, 1525, 2nd Edition, 1538, Hieronymus Formschneyder, Nürenberg, cited after A. Peltzer (Ed.), Albrecht Dürer'sUnterweisung der Messung, Munich, 1908.

7 B. Grünbaum et S. C. Geoffrey, "Tilings by regular polygons", Mathematics Magazine 50 (5), 1977, 227–247.

8 H. Steinhaus, *Mathematical snapshots,* Oxford University Press, New York, 1950. D. P. Chavey, "Periodic tilings and tilings by regular polygons. I: Bounds on the number of orbits of vertices, edges and tiles", *Mitteilungen aus dem Mathematischen Seminar Giessen* 164, 1984, 37–50.

9 G. Shawcross, "Periodic and Non-Periodic Tiling", 2012. https://grahamshawcross.com/2012/10/12/periodic-and-non-periodic-tiling/

are obtained. If the tiling is such that periodic tiling cannot be obtained even within local patches, the tiling is called aperiodic. One of the best known among them is Penrose tiling.[10] Aperiodic tilings are still among the most challenging geometric problems of tiling the plane.

TILE SHAPES AND TILING SYMMETRIES

If we start from the shape of the tiles that can participate in the tiling, we return to the regular tilings made up by the congruent tiles. They are tiled by one of the following three regular polygons: equilateral triangle, square and regular hexagon. Triangular tiling (*deltille*)[11] consists of equilateral triangles, more precisely, 6 of them organized around the same vertex. Square tiling (*quadrille*) consists of congruent squares, 4 of them around the same vertex. Hexagonal tiling (*hextille*) is formed of 3 hexagons around a common vertex. The interior angles of these polygons multiplied by an integer, produce a full circle, 2π, which is not the case with other polygons. Moreover, the angular compliance of two different polygons is also a problem due to the sum of their interior angles. According to Grünbaum[12] and Shepard, there are only 5 regular polygons that can tile a whole plane. These are: triangle, square, hexagon, octagon and dodecagon, wherein the octagon appears in one case of Archimedes' tiling only. Pentagon is not present in this sequence, since its interior angles do not fulfill the above condition.

Periodic tiling can also have rotational symmetry – if the prototiles or the fundamental region is rotated by an angle of $2\pi/n$ exactly n times. Then we get a pattern, i.e. tiling with n-fold symmetry. For example, an equilateral triangle has 3-fold symmetry, but a triangular tiling has 6-fold symmetry, because we have to rotate a triangle 6 times by 60° to fill the full circle. A square has 4-fold symmetry and so does the square tiling. Hexagon has 6-fold symmetry, but hexagonal tiling has 3-fold symmetry, etc. However, although the pentagon itself has 5-fold symmetry, tiling that uses solely regular polygons and has 5-fold symmetry does not exist. But are there other solutions?

Attempts to solve the 5-fold symmetrical tiling date even before Kepler's studies. A century earlier, Albrecht Dürer gave in his treatise *Unterweysung der messung mit dem Zirckel und Richtscheyt*,[13] among other solutions, his periodic and non-periodic tilings with pentagons and rhombuses ("diamonds"). In fact, he gave an unfinished scheme for radial tiling (Fig. 1a), where we can see *"fivefold nucleus, which can be extended to a multiple twin of fivefold symmetry"*.[14] Anyhow, his construction of radial pentagonal tiling remains the one of the ineluctable solutions of using pentagon in tiling the plane, while respecting 5-fold symmetry. Kepler, on the other hand, gives his famous solution in *Harmonices Mundi*, a tiling with decagons, pentagons and

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10 R. Penrose, "The role of aesthetics in pure and applied mathematical research", *Bulletin of the Institute of Mathematics and Its Applications*10, 1974, 266–271.

11 J. H. Conway, "The Orbifold Notation for Surface Groups", In: Groups, Combinatorics and Geometry, eds. M. W. Liebecket J. Saxl, ch. 36, Cambridge [England]; New York: Cambridge University Press, 1992, 438–447.

12 B. Grünbaum et S. C. Geoffrey, "Tilings by regular polygons", Mathematics Magazine 50 (5), 1977, 227–247.

13 A. Dürer, *Unterweysung der messungmitdem Zirckel und Richtscheyt*, 1st Edition, 1525, 2nd Edition, 1538, Hieronymus Formschneyder, Nürenberg, cited after A. Peltzer (Ed.), Albrecht Dürer'sUnterweisung der Messung, Munich, 1908.

14 L. Reinhard, "Dürer–Kepler–Penrose, the development of pentagon tilings" *Materials Science and Engineering* A 294, 2000, 263–267.

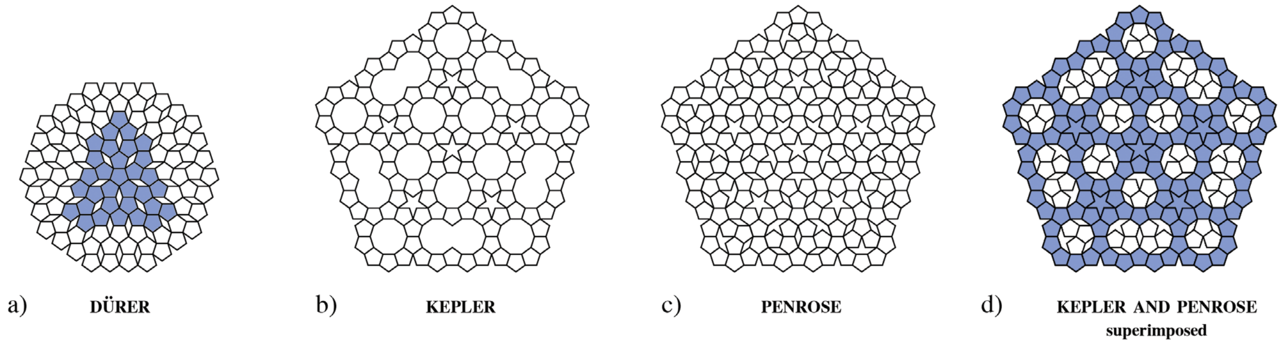

Fig. 1

fivefold stars, where the decagons partially overlap (Fig. 1b). Kepler named such an irregular tile a "*monstrum*". Symmetrically, however, this solution satisfies the tiling of the entire Euclidean plane. It will turn out to be visionary; not only does it meet the geometrical conditions of 5-fold symmetry, but it can also be multiplied in the manner of aperiodic tiling.

This problem was not clearly posed and defined until modern times. Aperiodic tilings are associated with the conjuncture of Hao Wang,[15] which states that "*if a set of tiles can tile the plane, then they can always be arranged to do so periodically*". His student, Robert Berger, refuted this conjecture by proving that aperiodic tilings exist.[16] A number of scientists then tried to determine the minimum number of different tiles needed to perform aperiodic tiling using Wangs tiles.[17] The solutions ranged from 104 tiles, given by Berger himself, to one given by Raphael M. Robinson[18] with 6 of them. Finally, Roger Penrose[19] reduces this number to two. Penrose gave his solutions with aperiodic tilings that give 5-fold symmetry, named after him: Penrose tilings, and the tiles used are named Penrose tiles. There is an infinite number of arrangements of these tiles that give Penrose tilings with local 5-fold symmetry, but only two of them are really 5-fold.[20] In all these solutions, we will see that the search for solutions with regular polygons has been abandoned, while other shapes that satisfy the conditions of symmetry and aperiodicity are allowed.

There are three basic types of Penrose tilings, according to the shape of the tiles.

a) P1: original Penrose tiling consisting of four different shapes: pentagon, five point star, "boat" and "diamond" (Fig. 1c).

b) P2: with two tile shapes: "kite" and "dart".

c) P3: with two shapes, "thin" and "thick" rhombus (v. Fig. 3c)

What is interesting is that Penrose tiling and Kepler's tiling **(**Fig. 1d**)** are strongly related,[21] i.e. that we can overlap them and notice the appearance of the same decagons, pentagons, five pointed stars and "monstrum" polygons on both tilings.

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15 H. Wang, "Proving Theorems by Pattern Recognition II," *Bell Systems Technical Journal* 40, 1961, 1–41.

16 R. Berger, "The Undecidability of the Domino Problem," *Memoirs of the American Mathematical Society* 66, 1966, 1–72.

17 Wang's tiles are square tiles, diagonally subdivided into 4 sections, each colored by one of the colors from the set.

18 R. M. Robinson, "Undecidability and Nonperiodicity for Tilings of the Plane," *InventionesMathematicæ* 12 (3), 1971, 177–209.

19 R. Penrose, "The role of aesthetics in pure and applied mathematical research", *Bulletin of the Institute of Mathematics and Its Applications* 10, 1974, 266–271

20 Data found in more different sources, and rely on the process of deflation.

21 L. Reinhard, "Dürer–Kepler–Penrose, the development of pentagon tilings", *Materials Science and Engineering* A 294, 2000, 263–267.

Also, the connection of Penrose tiling with quasicrystals need to be mentioned, as a 3D counterpart of the 2D tiling problem, which is close to the logic of the procedure that we present in this study.

The solution we provide, guided by the aforementioned knowledge, offers a non-periodic, radial tiling solution with 5-fold symmetry of the plane sector (region) with the central angle of 72°, consisting of a periodic arrangement of two shapes: a rectangle and an isosceles triangle. The inspiration for our research came from a spatial problem – 3D array of a specific group of polyhedral surfaces: *CC-II-5m*, so we also provide a spatial interpretation of the presented tiling.

METHOD AND LOGIC OF THE PROCEDURE

In this paper, we used the methods of descriptive geometry (orthogonal projection), 2D transformations, trigonometry and CAD. Although these problems are nowadays solved by algorithms and procedural graphics, the classical method – the one used by Dürer and Kepler still works well. Aided by precisely generated polygons via graphics software, construction and precision are no longer a problem. Furthermore, this method allows a greater author'sinfluence on the solution, including creativity, artistic twistsand atypical geometric solutions.

SPATIAL UNDERLAY OF THE PROPOSED RADIAL 5-FOLD TILING SOLUTION

To explain the origin of the solution given in this paper, we will refer to one specific group of polyhedral surfaces – concave cupolae of the second sort, because by using their geometry we came to the findings in this research.

Concave cupolae of the second sort[22] (abbr. *CC-II-n*) are polyhedra which, analogous to convex, Johnson cupolae[23] (J3, J4 and J5), have *{n}* and *{2n}* regular polygons as their bases in parallel planes. In the lateral surface, however, they are connected by equilateral triangles, diverse from Johnson solids where triangles and squares alternate. The arrangement of these triangles is such that they are set in two rows, forming *n* spatial open hexahedral cells. Arranged by a polar array so to fill a full circle, they give a concave lateral surface. Therefore, since all faces of the obtained polyhedral surface are regular polygons, all the edges are of equal length, *a*.

Unlike the convex cupolae with only three representatives, there are as many as 13[24] of the *CC-II-n*s. They can be obtained with any regular polygon in the range from {4} to {10} as a starting base. Also, they can be formed in two varieties depending on the way their planar net is folded: major, with greater height (*CC-II-n.M*) and minor, with lesser (*CC-II-n.m*). Of these 7 bases and 2 possible types for each *CC-II-n*, we exclude *CC-II-10.m* whose lateral vertices penetrate the icosagonal base. The properties of the geometry of these solids are described in detail in the dissertation "*Constructive – Geometric Elaboration of Toroidal Deltahedra with Regular Polygonal Base*" (*Konstruktivno – geometrijska obrada toroidnih deltaedara sa pravilnom poli-*

22 M. Obradović et S. Mišić, "Concave Regular Faced Cupolae of Second Sort", In: Proceedings of 13th ICGG, Dresden, August 2008, ed. G. Weiss, El. Book, Dresden, 2008, 1–10.

23 N. W. Johnson, Convex Solids with Regular Faces. Canadian Journal of mathematics 18 (1), 1966, 169–200.

24 In the number of earlier papers by the same authors, the number 14 is mentioned, but it refers to the possibility of forming a lateral surface in all 14 representatives, so that the possibility of elongated *CC-II-10*.m is allowed, although this one cannot exist with a surface surrounding continuous space in its basic form.

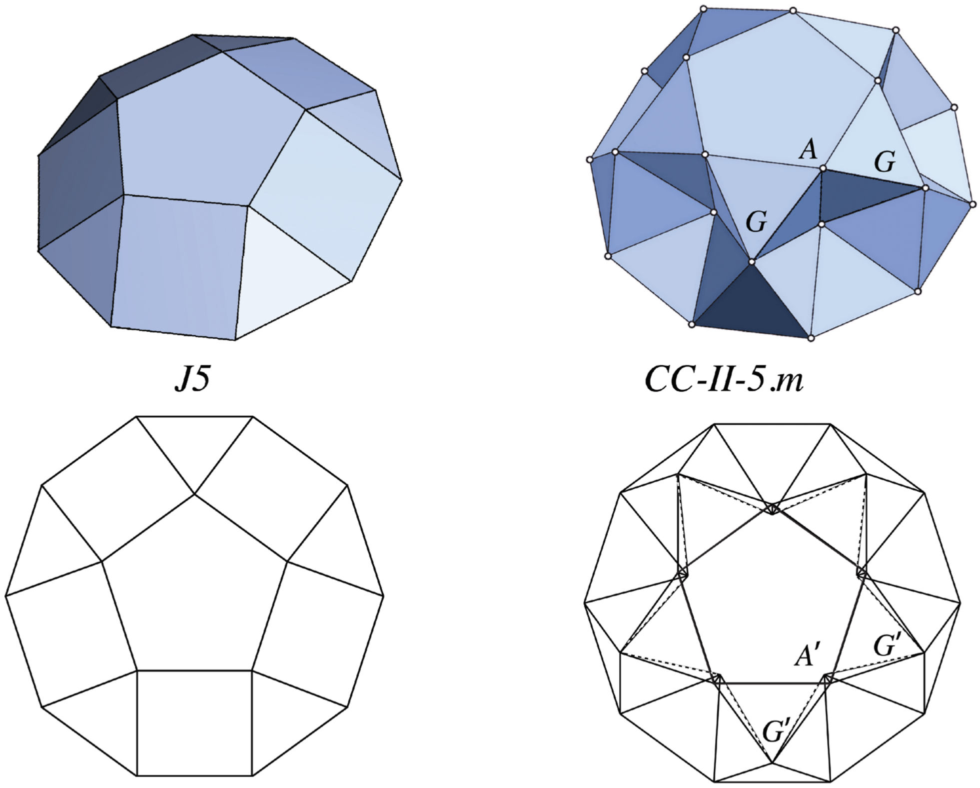


Fig. 2

gonalnom osnovom)[25] by M. Obradović (2006), although they are not considered as individual solids there, but as part of the outer deltahedral surface of the toroidal deltahedra. They are introduced as independent solids in the paper[26] by the same authors in 2008. One of these solids, a pentagonal concave cupola of the second sort, minor type (*CC-II-5.m* hereinafter) served as the basis for this study (Fig. 2).

The connection between *CC-II-n* and Euclidean tiling was considered in one of the earlier studies by the same author,[27] where the ways of subdividing the triangular faces of the *CC-II-n*'s lateral surface into triangles and hexagons were examined. Further research into the possibility of applying such tilings in the context of architectural design[28] led to some suggestions on the introduction of colors and different materials for these purposes. This paper goes a step beyond and instead of dividing the faces (subdivision method, and consequently substitution tiling), we now give a proposal for tiling by projecting a 3D polyhedral structure onto a 2D plane, similar to the "cut and project" method.

25 M. Obradović, *Konstruktivno-geometrijska obrada toroidnih deltaedara sa pravilnom poligonalnom osnovom, Constructive-geometrical elaboration on toroidal deltahedra with regular polygonal bases* (PhD thesis), University of Belgrade, Faculty of Architecture, 2006.

26 M. Obradović et S. Mišić, "Concave Regular Faced Cupolae of Second Sort", In: Proceedings of 13th ICGG, Dresden, August 2008, ed. G. Weiss, El. Book, Dresden, 2008, 1–10.

27 M. Obradović, "Tiling the Lateral Surface of the Concave Cupolae of the Second Sort", *Nexus Network Journal* 21/1, 2019, 59–77.

28 M. Obradovic, "Geometric Redesign of the Subdivided Surface of CC II: Application in Architecture", *Journal of Industrial Design and Engineering Graphics* 14, no. 1 (2019): 79–84.

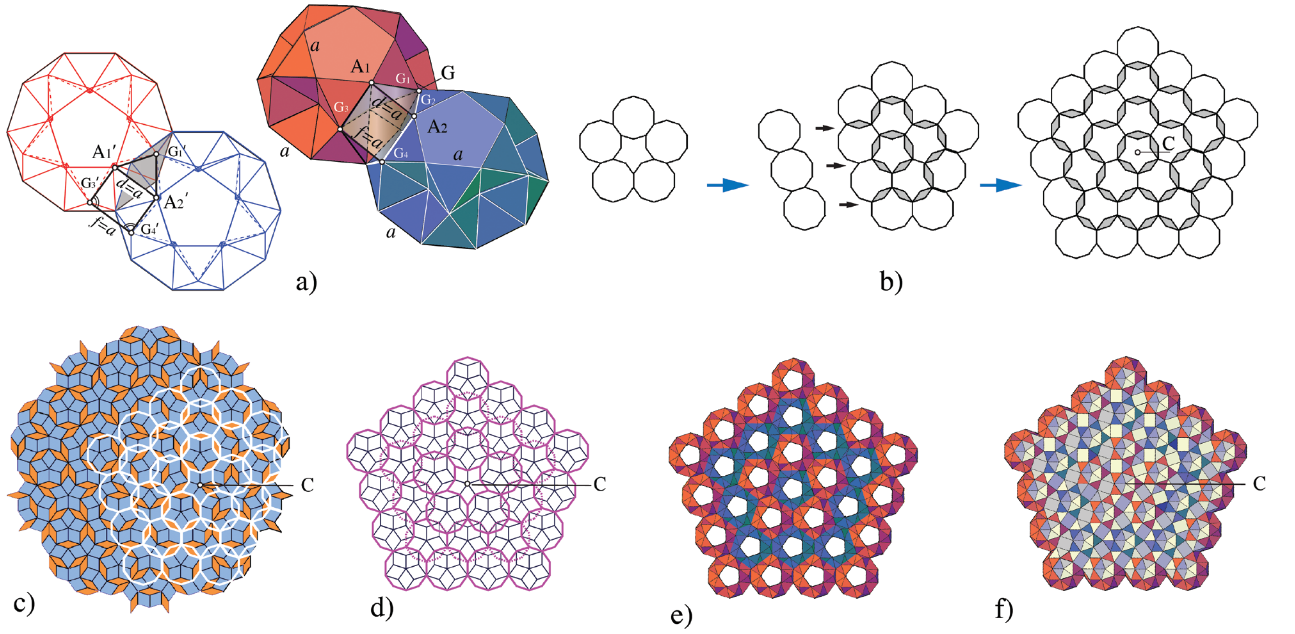


Fig. 3

DESCRIPTION OF THE PROCEDURE

If we adopt two *CC-II-5.m*s, translationally shifted so their bases are coplanar and their decagonal bases overlap sharing exactly two common vertices, the central vertices G_1 and G_2 of their open hexahedral cells touch, forming a common vertex *G* (Fig. 3a). This is easily trigonometrically provable on the basis of linear and angular parameters *CC-II-5.m*, which can be seen in: Obradović, 2006.[29] The overlapping areas of the decagon form precisely the Penrose tile: thin rhombus.

We then adopted a grid of decagons connected to each other by a common edge in concentric pentagonal "rings", as in Fig. 3b, so that the decagons from the adjacent rings overlap in the described manner. This grid outlines the regular pentagon. It can also be seen locally in Penrose tiling, but more broadly, it will not be respected the further we move away from the centroid *C* of the central five-pointed star (Fig. 3c). In fact, Penrose tiles, "thick" and "thin" rhombus, can fit into this grid, albeit precisely in every second "ring", while those in between are partially trimmed (Fig. 3d). Five "thick rhombuses" form a new 5-pointed star prototile, translated in triangular number sequence within a section of 72°. The gaps are filled with "thin rhombuses". Followed by rotational symmetry, a radial tiling is obtained.

Now, we place the *CC-II-5.ms*' decagonal bases onto the decagons in the grid (Fig. 3e). The lateral surfaces of *CC-II-5.ms* within the same "ring" share the edges of the lower base, while the faces overlap with the ones in the adjacent rings. In this way, we get a complex polyhedral structure. Observed from the exterior – a composite, corrugated polyhedral surface (Fig. 3e). Instead of all regular faces, as is the case with *CC-II-5.m*, there will appear irregular polygons, obtained by the intersection of lateral faces in the lower row (closer to the decagonal base). To solve this, we will cover the deep folds between two adjacent cupolae with new polygons.

29 M. Obradović, *Konstruktivno – geometrijska obrada toroidnih deltaedara sa pravilnom poligonalnom osnovom / Constructive – geometrical elaboration on toroidal deltahedra with regular polygonal bases* (PhD thesis), University of Belgrade, Faculty of Architecture, 2006, 333, 399 and 400.

When we connect the two nearest vertices, A_1 and A_2 of the upper bases of the two adjacent *CC-II-5.m*s, we get a horizontal line segment (since both vertices are at the same height) whose length *d* exactly equals to the edge *a* of the cupola itself (Fig. 3a). This results from the geometry of the *CC-II-5.m*, and its linear and angular parameters.[30] If the distance

$$d = a = \overline{A_1A_2} \tag{1}$$

then the edges A_1G and A_2G of the adjacent *CC-II-5.m* that share the common vertex *G*, together with the edge *d* form an equilateral triangle A_1GA_2. On the other hand, the two other lateral edges connecting the observed vertices A_1 and A_2 with the succeeding vertices G_3 and G_4 of the adjacent *CC-II-5.m* from the same ring, will be parallel. This also results from the trigonometric properties of *CC-II-5.m* implying that the distance *f* between the vertices G_3 and G_4 is also equal to *a*.

$$f = a = \overline{G_3G_4} \tag{2}$$

The vertices A_1 and A_2 with the vertices G_3 and G_4 define the sides of the quadrilateral which are equal to *a*, while their all angles are right, which can also be easily proved trigonometrically.

This will be confirmed through the orthogonal projection of the newly created 3D polyhedral surface that covers the folds between the adjacent faces of the *CC-II-5.m*s, as explained below.

Thus, when we connect all the corresponding vertices of the *CC-II-5.m*s by the congruent line segments *a* and thus define polygonal faces of the new surface, we get a 3D covering, i.e. polyhedral surface consisting exclusively of regular polygons: triangles, squares and inherited pentagons of the upper bases. To reduce these three shapes to two, we will convert all the pentagons into a set of five equilateral triangles. We can place them above the pentagons, for augmentations, or beneath them, for incavations of the cupolae. Whichever position we choose, it will not affect the orthogonal projection, i.e. 2D tiling, as either way the pentagonal pyramids will be projected into an identical image. (Fig. 3f). It will only affect the appearance of the 3D covering.

This kind of surface can spread to infinity.

PROJECTION OF 3D COVERING AND FORMATION OF TILING

To get a Euclidean tiling, which means an arrangement of tiles that are all coplanar, we project the previously described 3D covering consisting of equilateral triangles and squares as faces, orthogonally onto the plane of *CC-II-5.m*s' bases. As a result, we get a tiling whose local region, with the center in the vertex *C*, has the shape of a regular pentagon, while the rows of tiles are strung radially in relation to the center *C*. Its prototiles are projections of the faces of the 3D covering (Fig. 4a). Each equilateral triangle is projected into an isosceles triangle of base side *a* and legs *b*, while the square will be projected into a rectangle of the same sides. (Fig. 4b). The ratio of these sides' lengths can be easily found using the projections of triangles – the

30 We do not give trigonometric proof in this paper, because it is elementary. It follows from the data given in the source mentioned in the footnote No. 25: that the distance from the projection G' of the vertex G to the edge of the upper base is equal to 0.688191, which was obtained algorithmically, by an iterative method. This exactly equals to the value of the radius of the circle inscribed in the regular pentagon, r=a/2tg36°=0.688191. The further is easily provable.

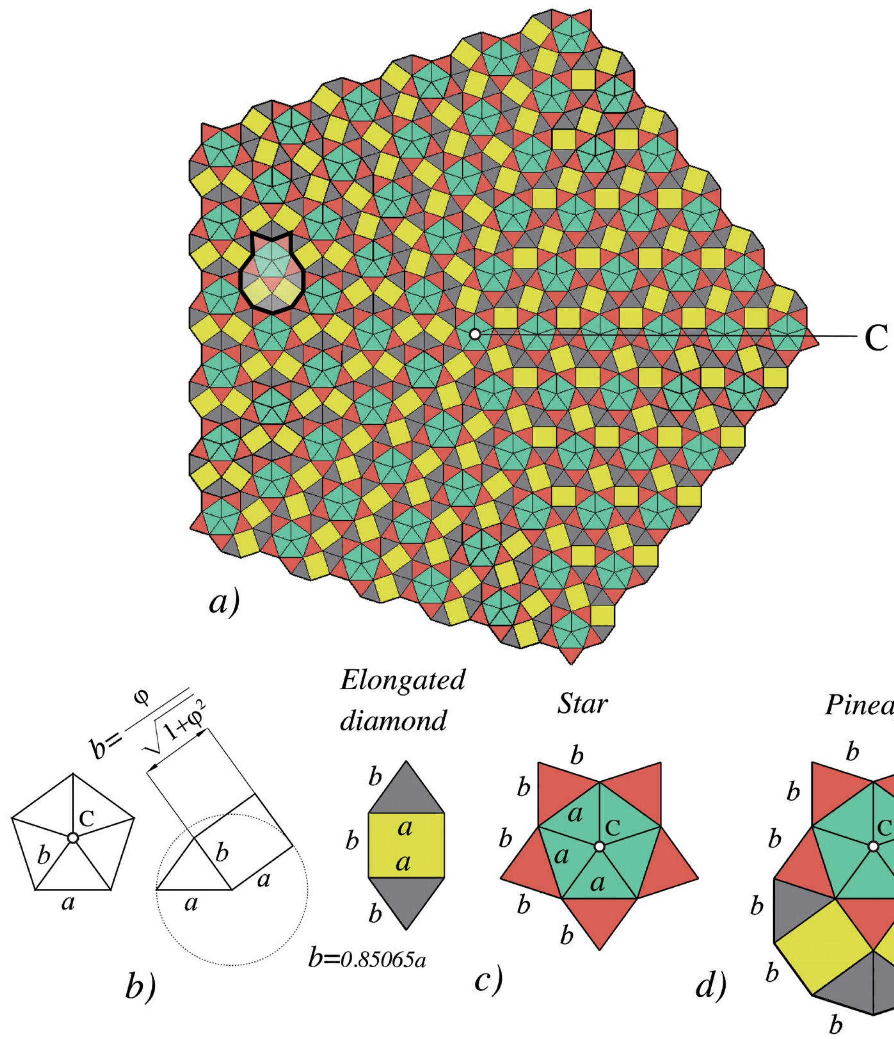


Fig. 4

lateral faces of regular-faced pentagonal pyramids by which we augment (or incavate) pentagonal bases of *CC-II-5.ms*. These equilateral triangles are all projected into congruent isosceles triangles corresponding to those obtained by the division of a regular pentagon into 5 segments, with a common central vertex *C* in the centroid of the pentagon (Fig. 4a). If *a* is the edge of *CC-II-5.m* parallel to the projection plane, we see it in the orthogonal projection as the side *a* of the tiling, undeformed in length. Thereby, as deformed, we see the one inclined towards the projection plane. Its projected length corresponds to the side *b*. The ratio of these sides is:

$$a : b = 1 : 0.85065 \tag{3}$$

$$b = \frac{\varphi}{\sqrt{1+\varphi^2}} \tag{4}$$

where φ is the golden ratio.

In the projection, these tiles are arranged so to form 5 different vertex figures (actually 7, but two of them are chiral) which are repeated by translation. We also notice that the initial prototiles are grouped into several new shapes. These are:

- 5-pointed star, obtained as an elevated pentagon (for simplicity, we named it a "star");
- elongated diamond (for simplicity, we named it a "diamond"), formed of two reflexive triangles, between which the rectangle is placed (Fig. 4c).

These shapes always appear in a full form, so they can be adopted as new prototiles whose all sides are *b*. Further, the “star” and the “diamond” fit together into a new shape which can be used as a single tile to perform the tiling. In this case, the exception of a single pentagonal patch that is the central “star” prototile, keeps this tiling from being monohedral. This shape also has all the contour sides of the length *b*, which enables it to fit with the abovementioned prototiles (Fig. 4d). Interestingly, the same contour is discovered in studies of quasicrystals. A group of Chinese authors in their study from 2017[31] named this shape “shield-like”, but it was not connected with a division into triangles and rectangles. In our study, we named this shape “pineapple“ because when dissected into smaller tiles, it resembles a stylized depiction of this fruit.

The “pineapple” shape can actually play a role of the fundamental region, in the periodic tiling solution, but this would be the subject of another research study. This shape can always be “broken” into simpler initial shapes.

DESCRIPTIVE-GEOMETRIC PROOF OF THE 3D COVERING FACES REGULARITY

We have previously explained that the distance between vertices A_1 and A_2 of the *CC-II-5.m*s in adjacent rings is equal to *a*, but the question remains: what are the distances between vertices of the adjacent *CC-II-5.m*s within the same ring? In reverse steps, from the 2D projection, i.e. tiling, back to the 3D covering, we will clarify this as well.

Let us tile the whole plane with the triangular and rectangular tiles as explained (v. Fig. 4a), and confirm that there are no gaps or overlaps. We see that only two line lengths appear in this tiling: *a* and *b*. Knowing that they originated as orthogonal projections of edges from a 3D corrugated polyhedral surface, we will ask the question of their lengths in space and are they all equal. We start from the height differences (Δh) of these lines’ extreme vertices. The edges of the length *a* are all horizontal, and their Δh = 0. Knowing that the edge length of the *CC-II-5.m* itself is seen in its real size *a*, in the basic pentagon (e.g. A_1A_1), then all the lines *a* in the tiling represent the horizontal line of the same length *a*, in space. Thus, isosceles triangles with base *a* and legs *b* actually represent equilateral triangles in space, rotated around the side *a*.

The lines of length *b* also have all equal height differences $\Delta h = h_A - h_G$. Since for some of them, which are the edges of *CC-II-5.m*s, we know for sure that their length is *a*, it follows that all other edges seen as *b* actually are of the length *a* in space. The right angles of the rectangle are formed by the sides *a* and *b*, where *a* is seen in the real size. If one of the legs is seen in its real size, this is a sufficient condition for a right angle to be projected undeformed, so the rectangle’s angles represent undeformed right angles in space. This means that the rectangle, in fact, is a projected square.

With this, we confirm that all the line segments that connect vertices *A* and *G* of the adjacent cupolae, mutually or crosswise, are equal and of the length *a*. Thereby the right angles are seen as undeformed and thus it follows that 3D corrugated covering consists only of equilateral triangles and squares.

Some other tilings ideas using the same tiles

31 Li, H et al., “Shield-like tile and its application to the decagonal quasicrystal-related structures in Al-Cr-Fe-Si alloys”, *Journal of Alloys and Compounds* 701, 2017, 494–498.

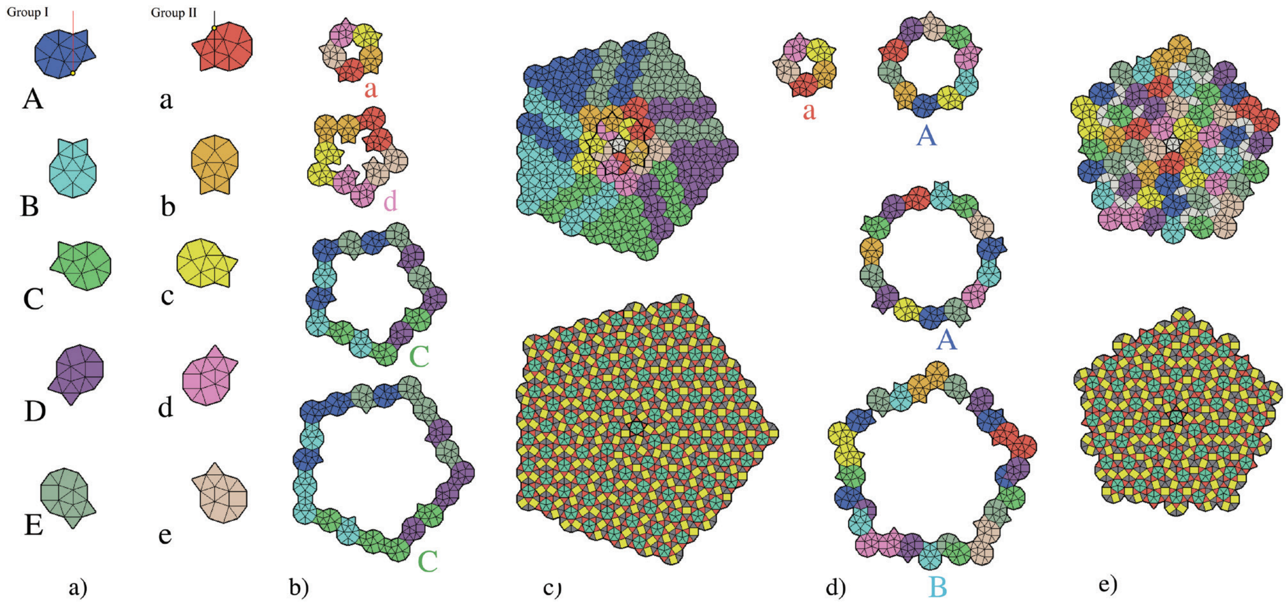


Fig. 5

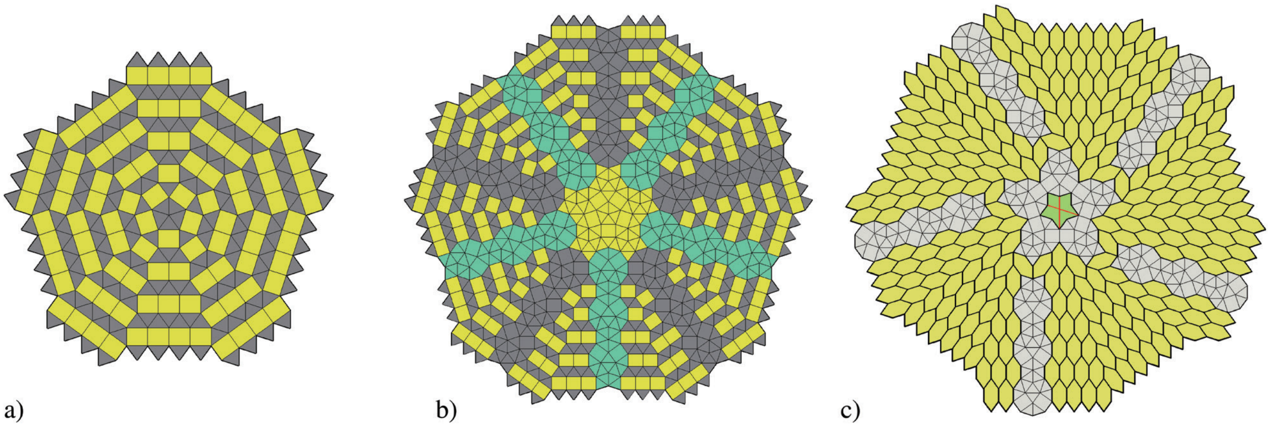
As with Penrose tilings, but also with many *k*-uniform ones, a single set of prototiles can form multiple different solutions. In the case of Penrose tiling, there is an infinite number of them. Since the geometry of pentagon is the basis of both Penrose tiling and the tiling given in this study, given that the adopted decagonal grid can also comprise Penrose tiles, it is expected that the proposed triangular and rectangular set of tiles can also provide an infinite number of solutions, some of which may be aperiodic. However, the exact substantiation of this conjecture and possible evidence awaits some future research.

In this paper, we will inspect only a few more solutions that have the central point – the vertex *C* in common, together with the property to spread radially from this point to infinity.

In Fig. 5a, we see the possible positions of "pineapple" tiles in a tiling. Tiles in which the "star" prototile is placed with the vertex point on its vertical axis of symmetry "down", are designated by capital letters: A, B, C, D and E and we denoted them as Group I. They differ in the rotation angle, so that each is rotated by 72° compared to the previous one. Tiles in which the "star" prototile is placed with the point on the vertical axis "up", rotated by 36° in relation to the previous cases, are designated by lowercase letters: a, b, c, d and e. We denoted them as Group II.

Figure 5b shows the stages of forming a radial tiling where 5 tiles of Group II: a, b, c, d, and e are set in the first "ring" around the central "star" prototile. In the second ring, 10 of these tiles are arranged in the order: d, d, e, e, a, a, b, b, c, c. When placed over the first ring with a common center *C*, the "star" tiles of both rings overlap[32] and coincide. In the third ring, 15 tiles of Group I alternate in the order: C, D, C, D, E, D, E, A, E, A, B, A, B, C, B. In the fourth ring, the 20 tiles of Group I alternate in order: C, C, D, C, D, D, E, D, E, E, A, E, A, A, B, A, B, B, C, B. The following rings translationally repeat the order of the tiles of Group I from the previous two rows.

32 We allow this overlapping because it produces the exact match of the smaller prototiles ("diamonds", triangles and rectangles, edge to edge) so that we can ignore the initial "pineaple" tile as fundamental region, since it only helped us in the organization of the tiles. Then we observe the resulting tiling as composed of triangles and rectangles.



a) b) c)

Fig. 6

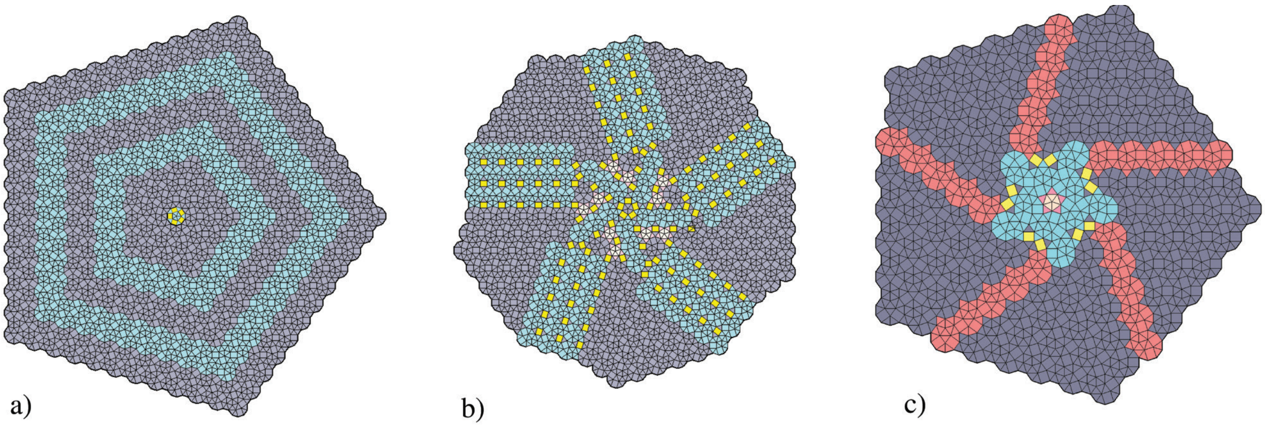
When these rings are laid together with the common center *C*, the result is tiling which looks very similar to the one originally presented (Fig. 4a), when we observe the solution with triangular and rectangular tiles, but in fact, their composition is more complex, as we see in Fig. 4c. There is a “twist” with 10 tiles of the Group II in the central region, and there is also one more “twisted”, i.e. rotated row of tiles in each of the five sectors of the tiling.

Figure 5d shows another tiling formed by a combination of tiles from Groups I and II. The first ring, with the “star” prototile in the center is identical to the first ring from the previous example, with the tiles: a, b, c, d, e. The second ring comprises alternating tiles of both Groups I and II: A, c, B, d, C, e, D, a, E, b. In the third ring they are set in the arrangement: A, c, B, d, C, e, D, a, E, b, and then in the fourth ring the order of the tiles is: A, E, d, B, A, e, C, B, a, D, C, b, E, D, c. In the fifth, sixth and subsequent “rings”, the tiles of Group I and II also alternate. The gaps in between are filled with “diamond” tiles, which enables the fulfilment of the first tessellation rule: no gaps or overlaps, and the tiles are aligned edge to edge, but now it refers to the smaller prototiles: “star”, “diamond”, triangle and rectangle.

We can observe the grouping of tiles that build different figures, and in the final tiling we will notice lines, formed by “diamond” tiles, which create interesting geometric patterns over this tiling (Fig. 5e).

In addition to these examples, we will give 3 more simple variations. As we could see, “diamond” tiles have such a geometry that they contain both triangles and rectangles, so that they can independently form a tiling, without the participation of “star” prototiles. Such a tiling is shown in Fig. 6a. It is a radial tiling consisting of 10 identical sectors made up of a series of “diamond” tiles, which are arranged in the observed sector periodically. They are connected to each other in triangular number sequence. Two such sectors are adjoined by the edges and corresponding angles, wherein one is rotated by 36° and shifted by length *b*. This pair is then multiplied by rotational 5-point symmetry around the vertex of the first, protruding triangle, until the full circle is closed. Thus we get a radial tiling composed of the very same starting triangles and rectangles.

These sectors of locally periodic “diamond” tilings can be combined with “pineapple” tiles, so that they create other radial tilings composed of initially defined prototiles. As shown in Fig. 6b, in one such example, the “pineapple” tiles form stripes through the middle of the five sectors supplemented by diamonds, creating new radial tiling. In Fig. 6c we see the case of tiling which occurs when we shift the “pineapple” strip from the previous example by one “diamond” tile to the left. Then



a) b) c)

Fig. 1

we get “displaced” radial tiling, actually a rotational tiling with the center of rotation set also in point *C*. Now, pineapple tiles do not run along the directions that pass through the center of tiling, but pass by it. Obviously, all these tilings can go on indefinitely with a clear scheme of further sequence.

We also show some visually interesting solutions that share the same feature with the ones presented above: the central point from which the tiles radially spread. Fig. 7a shows the solution where “pineapple” tiles are connected without overlaps or gaps that need to be filled with “diamond” tiles. “Pineapple” tiles from the same group alternate in different rings around the central decagonal tile, the only “alien” tile between the “pineapples”. Thus, the rings with tiles from Group I are shown in gray, while the tiles from Group II are shown in cyan. The tiles can be arranged in single, double, triple and *n*-fold rings, so this tiling can also be called “concentric”. Such a feature gives potential in creating different design solutions.

The following example, given in Fig. 7b, we named “windmill” tiling, because the tiles from Group II are grouped into locally periodic regions resembling windmill blades.

The solution shown in Fig. 7c, resembles the solution from Fig. 6c, but with the rows of “diamond” tiles replaced by “pineapple” ones of Group I, while the first ring consists of “pineapple” tiles of Group II. However, it is actually the same solution as in Fig. 5e except that the rows of “pineapples” are now highlighted in color, so that they can be seen as stripes connected to the “lumps” of the core.

Certainly, the possibilities of forming multitude of such tiling, from periodic, through generative, non-periodic, and even aperiodic, are not exhausted by this and it will be the subject of further research.

APPLICATION OF THE PRESENTED TILINGS IN ARCHITECTURE AND APPLIED ART

Geometric shapes, as an inherent stamp of intelligent conception, are found both in decoration and design of human settlements and utility objects from the beginnings of civilization. This is evidenced by the earliest artifacts of the Prehistoric Period, through the primitive cultures of Africa and Australia, historical remains of lost civilizations and the existing ones, to the current artistic and architectural trends. Among them, regular polygons occupy a special place, not only as shapes of tiles for covering surfaces, but also as shapes of the ground planes of buildings and even of entire cities. From the ideas of the “ideal city”, dating back to Plato and his “Republic”, through the utopian cities of the Renaissance and the works of Leon

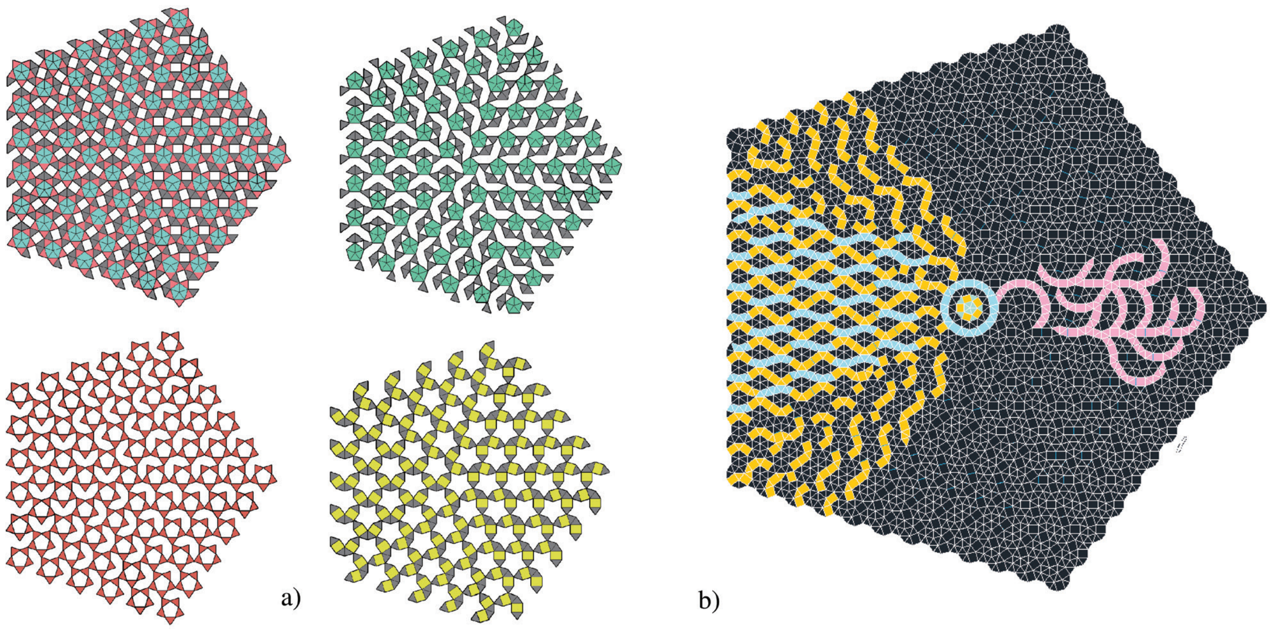


Fig. 8

Battista Alberti[33] (1404–1472), to military engineers such as Sébastien Le Prestre de Vauban (1633–1707) in Early Modern Period, who used a matrix of regular polygons to create invincible military fortifications,[34] regular polygons, as forms of "ideal geometry", have permeated architectural achievements to this day. We conclude that regular polygons, with their multiple symmetries, are immanent to the human experience of aesthetics, since we also find them in biomorphic structures such as flowers, honeycombs or crystals, which we associate with beautiful, pleasant or precious. Therefore, the regular pentagon, as one of them, served as an inspiring basic shape whose tiling we solve in this study. Two more regular polygons, equilateral triangle and square, appear in the 3D covering as faces that build a corrugated polyhedral surface. Choosing the regular polygons in design, we get a kind of artistic verification, as these forms are perceived as attractive and orderly.

The pentagon, as a geometrically fascinating shape with its 5-fold symmetry, becomes especially challenging when it comes to fitting with the rectangular matrix of the space we are used to, so it still represents a puzzle for both designers and contractors. That is why the solutions that enable its mastering, whether they concern tilings of surfaces or ideas for its division into simpler shapes are always useful. Penrose tiling is evidently one of them, but with its non-standard shape of the tiles, whether it be "thick" and "thin" rhombuses or "kites" and "darts", it requires knowledge and obeying the tiling rules. This implies an educated and very spatially intelligent craftsman, which is not always attainable.

In the solutions we present here, there are only two types of tiles: a rectangle – a shape easy to produce, cut and pack, and an isosceles triangle, another simple shape that can be easily cut and packed. On the site, they can be assembled into larger prototiles: "stars", "diamonds" or "pineapples", so their further arrangement is much easier.

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33 L. B. Alberti, *De re aedificatoria. On the art of building in ten books*, translated by J. Rykwert, R. Tavernor and N. Leach, Cambridge, Massachusetts: MIT Press, 1988.

34 M. Obradović et S. Mišić, "Are Vauban's Geometrical Principles Applied in the Petrovaradin Fortress?", *Nexus Network Journal* 16/3, 2014, 751–776.

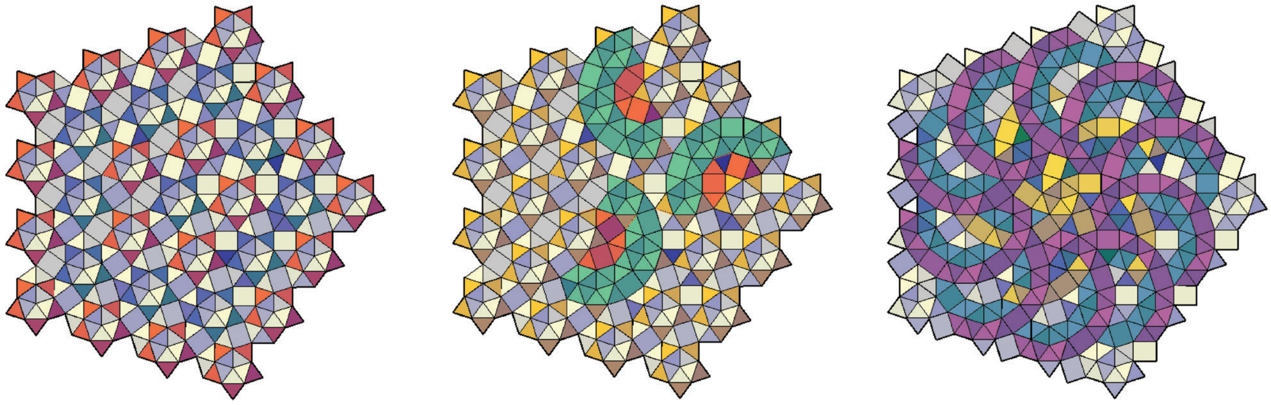

Fig. 9

Different arranging combinations of these tiles can provide a variety of solutions, from geometric to free-form, as in the classic mosaic. On the other hand, even if we observe a geometrically determined solution, the possibilities of various design interventions are practically unlimited. By applying different colors or materials, different patterns can be obtained, geometric or not, including even figuration.

We give several ideas for geometric (re)design of tiled pentagonal bases, according to the solutions given in the previous part of the paper.

1. By omitting some of the tiles (triangular, rectangular, or selected groups or regions) as in the solutions shown in Fig. 8a, we can get a new, hollow lattice structure that can be applied as brise soleil, room dividers, fences, pergolas or even rosettes in a modern interpretation of this classical, recognizable detail in sacral architecture.
2. By applying different colors, we can create patterns that can be used as decorative solutions for paving floors, walls, or as stained glass design (Fig. 8b).
3. The 3D corrugated polyhedral surface from which we started, also can be applied in various ways, e.g. in exterior, as a decorative facade cladding, or in the interior, as an acoustic cloud or wall panels. Also here, color intervention can give interesting design solutions (Fig. 9).

These proposals do not exhaust all the possibilities and ideas that can expand the creative approach to the application of tiling in architecture and applied arts. Let us remember the artist EmEmEm,[35] who makes “flacking” of cracks in the sidewalks, roadsides and walls on the streets of Lyon with a colorful mosaic. Artistic imagination and creativity will always find a way and a means to express the mselves, making our micro and macro spaces more pleasant and well-ordered places to live.

CONCLUSIONS

Starting with the problem of tiling the Euclidean plane with 5-fold symmetry, we have given a pentagon-based radial tiling solution that uses only two tiles: an isosceles triangle and a rectangle of sides *a* and *b*. We also presented several variations of the tiling. The starting point was the geometry of concave cupolae of the second sort of minor type, with the pentagonal base (*CC-II-5.m*). We proved that 3D covering composed of squares and equilateral triangles exactly fits the structure of the

35 N. Meek, “The French pavement street artist Ememem from Lyon going viral in Yorkshire”, *The Press*, 19/04/2021. https://www.yorkpress.co.uk/news/19244057.french-pavement-street-artist-lyon-going-viral-yorkshire/

*CC-II-5.m*s arranged into concentric decagonal rings. By projecting the resulting 3D covering onto the base plane of the *CC-II-5.m*s, we obtain 2D tiling with the center at the centroid *C* of the central pentagon of the initial polyhedral structure. It is a radial tiling with a locally periodic array of tiles, obtained by the rotational 5-fold symmetry. Tiles that participate in this tiling can be grouped into new prototiles: "star", "diamond" or "pineapple". With these tiles also, we can then cover the Euclidean plain without gaps or overlaps. There is a solution where only "diamond" tiles can be used for tiling. Similarly, the 2D solution given as a projection of 3D covering, can be obtained by applying the "pineapple" tile alone, with the single exception of the central "star" tile. Using these prototiles, we can create unlimited number of tilings, as with Penrose tiling.

This way of tiling gives us an advantage in covering pentagonal bases over Penrose tiling pattern in the sense that it is easier to perform, and that the tiles themselves are easier for production, cutting and packaging. Therefore their eventual application is simpler. Pentagonal bases and ground planes are still a challenge for tiling with the fewer shapes of tiles and as simple to produce and perform as possible. Hence, finding new solutions is still the goal to strive for. Further research will certainly go in the direction of examining other possibilities of tiling with these tiles: from periodic to aperiodic.

ACKNOWLEDGEMENTS

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ILLUSTRATIONS

Fig. 1: Pentagonal tilings and 5-fold symmetrical patterns: Dürer, Kepler and Penrose.
Пентагонало поплочавање и 5-тоструко симетрични обрасци: Дирер, Кеплер, Пенроуз.
Fig. 2: Cupolae with the pentagonal base: J5 and CC-II-5.m
Куполе са пентагоналним базисом: J5 и CC-II-5.m
3: The process of creating a 3D "covering" formed on the basis of the CC-II-5.m's geometry - Illustration under c was made based on: Wikimedia Commons, File: Penrose Tiling (Rhombi), author: Inductiveload
Поступак настанка 3Д прекривача формираног на основу геометрије CC-II-5.m
Илустрација под c је рађена на основу: Wikimedia Commons, File: Penrose Tiling (Rhombi), author: Inductiveload
4: Tiling obtained as an orthogonal projection of the 3D "covering" and the tiles that constitute it
Поплочавање настало као ортогонална пројекција 3Д прекривача и плочице које га чине
5: Procedure for forming tilings using "pineapple" tiles
Поступак формирања поплочавања коришћењем „ананас" плочица
6: Tiling solutions based on "diamond" and "pineapple" tiles
Решења поплочавања базирана на „дијамант" и „ананас" плочицама
7: Solutions based on "diamond" and "pineapple" tiles with color interventions
Решења уз интервенцију бојом базирана на „дијамант" и „ананас" плочицама
8: Tiling solutions with omission or repainting of certain tiles
Решења поплочавања са изостављањем или пребојавањем одређених плочица
9: 3D cover and several solutions with color intervention
3Д прекривач и неколико решења са интервенцијом бојама

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ABBREVIATIONS

CC-II-n – concave cupola of the second sort
CC-II-n.M – concave cupola of the second sort, major type
CC-II-n.m – concave cupola of the second sort, minor type
CC-II-5.m – concave cupola of the second sort, minor type

Марија Ђ. ОБРАДОВИЋ, Слободан Ж. МИШИЋ

РАДИЈАЛНО ПОПЛОЧАВАЊЕ ТРОУГЛОВИМА И ПРАВОУГАОНИЦИМА ЗАСНОВАНО НА ПЕНТАГОНУ И ЊЕГОВА ПРОСТОРНА ТУМАЧЕЊА

Резиме: У овом раду дато је решење радијалног поплочавања петоугаоне основе помоћу троуглова и правоугаоника. Ово решење има просторну интерпретацију у 3Д распореду једнакостраничних труглова и квадрата који чине просторни наборани „прекривач" и прате петоугаону шему. Њихов распоред је диктиран положајем десетоугаоника који се делимично преклапају, пратећи распоред концентричних петоугаоних прстенова. У простору, десетоугаоници представљају основе петоугаоних конкавних купола друге врсте, нижег типа (*CC-II-5.m*). Прекривајући наборе између страна такве полиедарске структуре полигонима, користимо једнакостраничне троуглове и квадрате, захваљујући тригонометријским својствима *CC-II-5.m*. Пројектовани на раван базиса *CC-II-5.m* они формирају 3Д поплочавање које се радијално шири у простору, почевши од централне тачке C пентагона унутар саме почетне петоугаоне основе. Једнакостранични труглови се сви пројектују у подударне једнакокраке труглове који одговарају онима које добијамо радијалном секцијом правилног петоугла на 5 делова. Квадрати се пројектују у правоугаонике. Однос страница и троуглова и правоугаоника износи: *a:b= 0,85065:*1 или: , где је златни однос. Ови троуглови и правоугаоници чине радијално поплочавање које се састоји од 5 сектора где се утврђени обрасци уклапања плочица периодично понављају локално, мада је само поплочавање не-периодично. Ово једноставно решење није једино које можемо извести тим истим равним ликовима. Неколико других решења дајемо у раду. Формирање шара и арабески оваквим радијалним поплочавањем посебно је занимљиво у пољу примењених уметности. Добијени обрасци могу се користити у дизајну розета, брисолеја, мозаика, витража, ограда, преградних паравана и слично.

Кључне речи: поплочавање, петоугао, петострука симетрија, троугао, правоугаоник.